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Indirect control of quantum systems via an accessor: pure coherent control without system excitation

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Abstract

A pure indirect control of quantum systems via a quantum accessor is investigated. In this control scheme, we do not apply any external classical excitation fields on the controlled system and we control a quantum system via a quantum accessor and classical control fields control the accessor only. Complete controllability is investigated for arbitrary finite-dimensional quantum systems and exemplified by two- and three-dimensional systems. The scheme exhibits some advantages; it uses less qubits in the accessor and does not depend on the energy-level structure of the controlled system.

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1. Introduction

Quantum control is a coherence-preserving manipulation of a quantum system, which enables a time evolution from an arbitrary initial state to an arbitrary target state [1–4]. It was first proposed by Huang *et al* [5] in 1983 and was mainly used to control chemical reaction in its early days [6]. Recently it has attracted much attention due to its connection to quantum information processing. Actually the universality of quantum logic gates can be understood from viewpoint of complete controllability in quantum control [7]. Conventional quantum control is the coherent control of quantum systems using classical external fields. Controllability of this *semi-classical* control is well studied [8], especially the complete controllability of finite-dimensional quantum systems using the Lie algebra method [9, 10] graph method [11] and transfer graph method [12]. The Lie algebra approach plays an important role in the investigation of both the classical control [13] and the quantum control.

In some circumstances in quantum information processing, there is need to control qubits using quantum controllers such as a quantum accessor and its environment. For example, in connection with the fundamental limit of quantum information processing and influence of decoherence to quantum control, we have proposed an indirect scheme for quantum control where the controller is also quantum [14]. To avoid switching the couplings between qubits, Zhou *et al* introduced the so-called encoded qubits to realize the universal quantum computation with local manipulation of physical qubits only [15]. Here the physical qubits do not involve the quantum computation and play the role of quantum controllers. Recently Hodges *et al* proposed an universal indirect control of nuclear spins using a single electron spin acting as an accessor driven by microwave irradiation of resolved anisotropic hyperfine [16], which has an important application for spin based solid state quantum information processing. Therefore the control of quantum systems using quantum controllers has significant application in quantum information processing and has attracted much attention recently. Authors of this paper proposed the conception of the *indirect control* of quantum systems where the quantum systems are controlled via a quantum accessor and the classical control fields control the accessor only [17]. Similar works were proposed in a different context [20] for spin-$1/2$ particles. Romano [18] and Pechen [19] considered the incoherent control induced by environment modeled as quantum radiation fields.

In our previous paper [17], we proposed a scheme for the control of an arbitrary finite-dimensional quantum system using a quantum accessor modeled as a qubit chain with XY-type neighborhood coupling. We find the conditions of way of coupling between the controlled system and the accessor and the minimal length of qubit chain to ensure the complete control of the controlled system. However, besides the classical control fields controlling the accessor, we also apply a constant classical field on the controlled system to excite the system through dipole interaction. Without the excitation field, the system is not completely controllable and for the two-dimensional case, underlying Lie algebra is the Symplectic algebra sp(4), rather than su(4). Another disadvantage of this scheme is that the controllability depends on the structure of energy levels of the controlled system. In this paper we shall remove the excitation field and propose a *pure* indirect control scheme where the external control fields control the accessor *only*. We shall see that, in comparison with the scheme in [17], the new scheme proposed in this paper exhibits some advantages besides the removal of excitation field, for example, it uses less qubits of the accessor for complete control of the controlled system and there are no particular requirements on the structure of the energy levels of the controlled system.

The remaining part of this paper is organized as follows: we formulate the control system without system excitation in section 2 and then introduce the *selection* operators and apply it to the study of controllability of a two energy level system in section 3. The case of a three-dimensional system is investigated in section 4. The general approach of controllability of indirect control of arbitrary finite-dimensional systems is investigated in section 5. We conclude in section 6.

2. Indirect control system

In this section we shall formulate the indirect control system and fix the notations we will use later on. Suppose that the system to be controlled is an N-dimensional quantum system described by the following Hamiltonian:

$$H_S = \sum_{i=1}^{N} E_i e_{ii} = \sum_{i=1}^{N-1} \epsilon_i h_i, \tag{1}$$

where E_i's are eigenenergy of the system, e_{ij} is an $N \times N$ matrix with matrix elements $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}, h_i = e_{ii} - e_{i+1,i+1}$ are Cartan generators of the Lie algebra su(N) and $\epsilon_i \equiv E_1 + E_2 + \cdots + E_i$. Here we have assumed that tr $H_S = 0$ without losing generality.

Note that we do not apply any external classical excitation field on the system S as we did in [17]. The excitation field, although it is a constant field, makes the indirect control in [17] not really pure indirect.

The quantum accessor is modeled as a qubit chain with XY-type neighborhood coupling

$$H_A = H_A^0 + H_A^I,$$
$$H_A^0 = \sum_{i=0}^{M} \hbar\omega_i\sigma_z^i, \qquad H_A^I = \sum_{i=1}^{M-1} c_i\sigma_x^i\sigma_x^{i+1}, \tag{2}$$

where $c_i \neq 0$ and

$$\sigma_\alpha^i = 1 \otimes \cdots \otimes 1 \otimes \sigma_\alpha \otimes 1 \otimes \cdots \otimes 1, \tag{3}$$

namely σ_α on the site i and 1 on any other sites.

The system and the accessor are coupled as

$$H_I = \sum_{\{\alpha_i\}} \left[\sum_{j=1}^{N-1} \sum_{k=0,\pm1} g_{(\alpha_i)}^{j(k)} s_j^k\right] \otimes \sigma_{\alpha_1}^1 \cdots \sigma_{\alpha_M}^M, \tag{4}$$

where $\{\alpha_i\} = \{\alpha_1, \alpha_2, \ldots, \alpha_M\}$ and each $\alpha_i = x, y, z$ rather than just x, y as in previous paper [17], s_j^k is defined as

$$s_j^k = \begin{cases} x_j & \text{when } k = 1; \\ h_j & \text{when } k = 0; \\ y_j & \text{when } k = -1, \end{cases} \tag{5}$$

and

$$x_j = e_{j,j+1} + e_{j+1,j}, \qquad y_j = \mathrm{i}(e_{j,j+1} - e_{j+1,j}), \tag{6}$$

along with h_j constitute the Chevalley basis of the Lie algebra su(N) [21].

It is known that when we remove the excitation field, the indirect system is not completely controllable if $\alpha_i = x, y$ only [17]. In the case of indirect control of a 2-level system, the Lie algebra is sp(4) with dimension 10, rather than the su(4) [17]. However, as the example we presented in [17], we can rotate the system to remove the excitation field, but as price paid the interaction Hamiltonian includes σ_z for the accessor part. So this is why we include $\alpha_i = z$ in the coupling Hamiltonian (4).

We suppose that we can control each qubit of the accessor completely through external classical fields. The complete control of each qubit in a qubit chain can be achieved via global manipulation [23, 24]. Therefore the total control system is

$$H = H_0 + \sum_{j=1}^{M} \left[f_j(t)\sigma_x^j + f_j'(t)\sigma_y^j\right],$$
$$H_0 = H_S + H_A + H_I, \tag{7}$$

where $f_j(t)$ and $f_j'(t)$ are two independent classical control fields.

In the rest of this paper we shall investigate the complete controllability of the indirect control scheme (7), namely in what conditions the system is completely controllable. More precisely, in what conditions the Lie algebra generated by the skew-Hermitian operators $\mathrm{i}H_0$, $\mathrm{i}1 \otimes \sigma_x^j$ and $\mathrm{i}1 \otimes \sigma_y^j$ $(j = 1, 2, \ldots, M)$ is su($2^M N$), or its dimension is $(2^M N)^2 - 1$.

3. Selection operators and indirect control of single qubit

To prove the complete controllability, we define the so-called *selection operators* S_{xy}^k and S_{yx}^k acting on the Pauli's operators of kth qubit of the accessor

$$S_{xy}^k = \tfrac{1}{4}\mathrm{ad}_{\mathrm{i}\sigma_x^k}\mathrm{ad}_{i\sigma_y^k}, \qquad S_{yx}^k = \tfrac{1}{4}\mathrm{ad}_{\mathrm{i}\sigma_y^k}\mathrm{ad}_{i\sigma_x^k} \tag{8}$$

where $\mathrm{ad}_{\mathrm{i}\sigma_x^k}$ is the adjoint representation of the Lie algebra su(2) of the kth qubit

$$\mathrm{ad}_X(Y) = [X, Y], \qquad \forall\, X, Y \in \mathrm{su}(2). \tag{9}$$

From definition (9), it follows

$$S_{xy}^k(*) = \tfrac{1}{4}\big[\mathrm{i}\sigma_x^k, \big[\mathrm{i}\sigma_y^k, *\big]\big], \qquad S_{yx}^k(*) = \tfrac{1}{4}\big[\mathrm{i}\sigma_y^k, \big[\mathrm{i}\sigma_x^k, *\big]\big]. \tag{10}$$

It is easy to prove that

$$S_{xy}^k\big(\mathrm{i}\sigma_\alpha^k\big) = \begin{cases}\mathrm{i}\sigma_y^k, & \alpha = x;\\ 0, & \alpha = y, z;\end{cases} \tag{11}$$

$$S_{yx}^k\big(\mathrm{i}\sigma_\alpha^k\big) = \begin{cases}\mathrm{i}\sigma_x^k, & \alpha = y;\\ 0, & \alpha = x, z;\end{cases} \tag{12}$$

namely, S_{xy}^k transforms $\mathrm{i}\sigma_x^k$ to $\mathrm{i}\sigma_y^k$ and annihilates any others, S_{yx}^k transforms $\mathrm{i}\sigma_y^k$ to $\mathrm{i}\sigma_x^k$ and annihilates any others. Or in other words, S_{xy}^k can *select* σ_x^k and change it to σ_y^k from any linear combination of Pauli's matrices.

Now we show how to use those operators in the investigation of complete controllability with the two-dimensional system as an example. Here both the system and the accessor are a single qubit. The Hamiltonian of the system and the accessor is as follows:

$$\begin{aligned} H_0 = \hbar\omega_S\sigma_z \otimes 1 + \hbar\omega_A(1\otimes\sigma_z) + (g_{xx}\sigma_x + g_{yx}\sigma_y + g_{zx}\sigma_z)\otimes\sigma_x \\ + (g_{xy}\sigma_x + g_{yy}\sigma_y + g_{zy}\sigma_z)\otimes\sigma_y + (g_{xz}\sigma_x + g_{yz}\sigma_y + g_{zz}\sigma_z)\otimes\sigma_z. \end{aligned} \tag{13}$$

We suppose we can control the accessor fully

$$H_C = f_1(t)1\otimes\sigma_x + f_2(t)1\otimes\sigma_y, \tag{14}$$

where $f_1(t)$ and $f_2(t)$ are two independent classical control fields to control the accessor. The Lie algebra generators are $\mathrm{i}H_0$, $\mathrm{i}1\otimes\sigma_x$ and $\mathrm{i}1\otimes\sigma_y$ and the generated Lie algebra is denoted by $\mathcal{L}$. It is obvious that

$$-2^{-1}[\mathrm{i}1\otimes\sigma_x, \mathrm{i}1\otimes\sigma_y] = \mathrm{i}1\otimes\sigma_z \in \mathcal{L}. \tag{15}$$

So we can subtract the second term in H_0 and obtain the Lie algebra element $\mathrm{i}H_0' \equiv \mathrm{i}H_0 - \mathrm{i}\hbar\omega_A(1\otimes\sigma_z) \in \mathcal{L}$.

Now we apply the selection operators on the element $\mathrm{i}H_0'$, yielding

$$S_{xy}(\mathrm{i}H_0') = \mathrm{i}(g_{xx}\sigma_x + g_{yx}\sigma_y + g_{zx}\sigma_z)\otimes\sigma_y \in \mathcal{L}, \tag{16}$$

$$S_{yx}(\mathrm{i}H_0') = \mathrm{i}(g_{xy}\sigma_x + g_{yy}\sigma_y + g_{zy}\sigma_z)\otimes\sigma_x \in \mathcal{L}. \tag{17}$$

In fact, by evaluating the commutation relation of (16), (17) with the generators σ_x and σ_y of accessor in (16) and (17) can be changed to any σ_α ($\alpha = x, y, z$).

We further subtract the terms (16) and (17) from $\mathrm{i}H_0'$ and then calculate its commutation relation with $\mathrm{i}1\otimes\sigma_y$. We find

$$\mathrm{i}(g_{xz}\sigma_x + g_{yz}\sigma_y + g_{zz}\sigma_z)\otimes\sigma_\alpha \in \mathcal{L}. \tag{18}$$

If the following condition

$$\det\begin{pmatrix} g_{xx} & g_{yx} & g_{zx} \\ g_{xy} & g_{yy} & g_{zy} \\ g_{xz} & g_{yz} & g_{zz} \end{pmatrix} \neq 0 \tag{19}$$

is satisfied, we find nine Lie algebra elements

$$\mathrm{i}\sigma_\alpha \otimes \sigma_\beta \in \mathcal{L}, \tag{20}$$

where $\alpha, \beta = x, y, z$. The condition (19) can be achieved by choosing, for example, $g_{xx} = g_{yy} = g_{zz} = 1$ and any other zero. As we already have $\mathrm{i}1 \otimes \sigma_\alpha \in \mathcal{L}$, so we only need to prove $\mathrm{i}\sigma_\alpha \otimes 1 \in \mathcal{L}$. For this purpose, we evaluate

$$-2^{-1}[\mathrm{i}\sigma_x \otimes \sigma_x, \mathrm{i}\sigma_y \otimes \sigma_x] = \mathrm{i}\sigma_z \otimes 1 \in \mathcal{L}, \tag{21}$$

$$2^{-1}[\mathrm{i}\sigma_x \otimes \sigma_x, \mathrm{i}\sigma_z \otimes \sigma_x] = \mathrm{i}\sigma_y \otimes 1 \in \mathcal{L}, \tag{22}$$

$$2^{-1}[\mathrm{i}\sigma_y \otimes 1, \mathrm{i}\sigma_z \otimes 1] = \mathrm{i}\sigma_x \otimes 1 \in \mathcal{L}. \tag{23}$$

In summary, the generated Lie algebra has fifteen generators $\mathrm{i}\sigma_\alpha \otimes \sigma_\beta$ where $\alpha, \beta = x, y, z, 0$ ($\sigma_0 \equiv 1$) and α, β cannot be 0 simultaneously, and they generate the Lie algebra su(4). Therefore the single qubit system is completely controllable under the condition (19).

4. Control of three-dimensional system

In this section we turn to the indirect control of the three-dimensional quantum system. The Hamiltonian takes the following form:

$$H_0 = H_S + H_A + H_{SA}$$

$$H_S = \sum_{i=1}^{3} E_i e_{ii} = E_1 h_1 + (E_1 + E_2) h_2$$

$$H_A = \hbar\omega_1 \sigma_z^1 + \hbar\omega_2 \sigma_z^2 + c\sigma_x^1 \sigma_x^2$$

$$H_{SA} = \sum_{\alpha,\beta=x,y,z} \left(g_{\alpha\beta}^{1(0)} h_1 + g_{\alpha\beta}^{2(0)} h_2 + g_{\alpha\beta}^{1(1)} x_1 + g_{\alpha\beta}^{2(1)} x_2 + g_{\alpha\beta}^{1(-1)} y_1 + g_{\alpha\beta}^{2(-1)} y_2\right) \otimes \sigma_\alpha^1 \sigma_\beta^2$$

where $h_1 = e_{11} - e_{22}$ and $h_2 = e_{22} - e_{33}$ are Cartan elements of Lie algebra su(3), and $x_i = e_{i,i+1} + e_{i+1,i}$ and $y_i = \mathrm{i}(e_{i,i+1} - e_{i+1,i})$ ($i = 1, 2$) are Chevelley basis of su(3) corresponding positive and negative simple roots, respectively. The complete control system is

$$H = H_0 + \sum_{k=1}^{2} \left(f_k(t) 1 \otimes \sigma_\alpha^k + f_k'(t) 1 \otimes \sigma_\alpha^k\right),$$

where $f_k(t)$ and $f_k'(t)$ are classical control fields.

It is easy to see that $\mathrm{i}1 \otimes \sigma_z^k \in \mathcal{L}$. So we can subtract the free Hamiltonian of the accessor form H_0 and obtain the following Lie algebra element:

$$H_0' = H_0 - \left(\hbar\omega_1 \sigma_z^1 + \hbar\omega_2 \sigma_z^2\right) \in \mathcal{L}. \tag{24}$$

It is easy to check that

$$S_{yx}^2 S_{yx}^1 (\mathrm{i}H_0') = \mathrm{i}\left(g_{yy}^{1(0)} h_1 + g_{yy}^{2(0)} h_2 + g_{yy}^{1(1)} x_1 + g_{yy}^{2(1)} x_2 + g_{yy}^{1(-1)} y_1 + g_{yy}^{2(-1)} y_2\right) \otimes \sigma_x^1 \sigma_x^2 \in \mathcal{L}, \tag{25}$$

$$S_{xy}^2 S_{yx}^1 (\mathrm{i}H_0') = \mathrm{i}\big(g_{yx}^{1(0)}h_1 + g_{yx}^{2(0)}h_2 + g_{yx}^{1(1)}x_1 + g_{yx}^{2(1)}x_2 + g_{yx}^{1(-1)}y_1 + g_{yx}^{2(-1)}y_2\big) \otimes \sigma_x^1\sigma_y^2 \in \mathcal{L}, \tag{26}$$

$$(1 - S_{xy}^2 - S_{yx}^2) S_{yx}^1 (\mathrm{i}H_0') = \mathrm{i}\big(g_{yz}^{1(0)}h_1 + g_{yz}^{2(0)}h_2 + g_{yz}^{1(1)}x_1 + g_{yz}^{2(1)}x_2 + g_{yz}^{1(-1)}y_1 + g_{yz}^{2(-1)}y_2\big) \otimes \sigma_x^1\sigma_z^2 \in \mathcal{L}. \tag{27}$$

After proper commutation with the external interaction Hamiltonian, we can change the accessor part in equations (25)–(27) to $\sigma_y^1\sigma_y^2$, $\sigma_y^1\sigma_x^2$ and $\sigma_y^1\sigma_z^2$, respectively. Then we subtract those Lie algebra elements from $\mathrm{i}H_0'$ and obtain the following Lie algebra element:

$$\mathrm{i}H_0'' = c\sigma_x^1\sigma_x^2 + \sum_{\alpha=x,z}\sum_{\beta=x,y,z} \big(g_{\alpha\beta}^{1(0)}h_1 + g_{\alpha\beta}^{2(0)}h_2 + g_{\alpha\beta}^{1(1)}x_1 + g_{\alpha\beta}^{2(1)}x_2 + g_{\alpha\beta}^{1(-1)}y_1 + g_{\alpha\beta}^{2(-1)}y_2\big) \otimes \sigma_\alpha^1\sigma_\beta^2. \tag{28}$$

To remove the term $c\sigma_x^1\sigma_x^2$ from $\mathrm{i}H_0''$, we evaluate the commutation relation between $\mathrm{i}H_0''$ and $\mathrm{i}\big(1\otimes\sigma_x^1\big)$, yielding

$$\mathrm{i}H_0''' = -2^{-1}\big[\mathrm{i}H_0'', \mathrm{i}1\otimes\sigma_x^1\big] = \mathrm{i}\sum_{\beta=x,y,z} \big(g_{z\beta}^{1(0)}h_1 + g_{z\beta}^{2(0)}h_2 + g_{z\beta}^{1(1)}x_1 + g_{z\beta}^{2(1)}x_2 + g_{z\beta}^{1(-1)}y_1 + g_{z\beta}^{2(-1)}y_2\big) \otimes \sigma_y^1\sigma_\beta^2 \in \mathcal{L}. \tag{29}$$

Then we can use the same trick as in (25)–(27) to prove

$$S_{yx}^2(\mathrm{i}H''') = \mathrm{i}\big(g_{zy}^{1(0)}h_1 + g_{zy}^{2(0)}h_2 + g_{zy}^{1(1)}x_1 + g_{zy}^{2(1)}x_2 + g_{zy}^{1(-1)}y_1 + g_{zy}^{2(-1)}y_2\big) \otimes \sigma_x^1\sigma_x^2 \in \mathcal{L}, \tag{30}$$

$$S_{xy}^2(\mathrm{i}H''') = \mathrm{i}\big(g_{zx}^{1(0)}h_1 + g_{zx}^{2(0)}h_2 + g_{zx}^{1(1)}x_1 + g_{zx}^{2(1)}x_2 + g_{zx}^{1(-1)}y_1 + g_{zx}^{2(-1)}y_2\big) \otimes \sigma_x^1\sigma_y^2 \in \mathcal{L}, \tag{31}$$

$$\big(1 - S_{xy}^2 - S_{yx}^2\big)(\mathrm{i}H''') = \mathrm{i}\big(g_{zz}^{1(0)}h_1 + g_{zz}^{2(0)}h_2 + g_{zz}^{1(1)}x_1 + g_{zz}^{2(1)}x_2 + g_{zz}^{1(-1)}y_1 + g_{zz}^{2(-1)}y_2\big) \otimes \sigma_x^1\sigma_z^2 \in \mathcal{L}. \tag{32}$$

Now we have found six independent Lie algebra elements equations (25)–(27) and equations (30)–(32), in which the accessor part can be changed to the same $\sigma_\alpha^1\sigma_\beta^2$ ($\alpha, \beta = x, y, z$) by evaluating proper commutation with the external interaction Hamiltonian. If the coefficients satisfy the following condition:

$$\det\begin{pmatrix} g_{yx}^{1(0)} & g_{yx}^{2(0)} & g_{yx}^{1(1)} & g_{yx}^{2(1)} & g_{yx}^{1(-1)} & g_{yx}^{2(-1)} \\ g_{yy}^{1(0)} & g_{yy}^{2(0)} & g_{yy}^{1(1)} & g_{yy}^{2(1)} & g_{yy}^{1(-1)} & g_{yy}^{2(-1)} \\ g_{yz}^{1(0)} & g_{yz}^{2(0)} & g_{yz}^{1(1)} & g_{yz}^{2(1)} & g_{yz}^{1(-1)} & g_{yz}^{2(-1)} \\ g_{zx}^{1(0)} & g_{zx}^{2(0)} & g_{zx}^{1(1)} & g_{zx}^{2(1)} & g_{zx}^{1(-1)} & g_{zx}^{2(-1)} \\ g_{zy}^{1(0)} & g_{zy}^{2(0)} & g_{zy}^{1(1)} & g_{zy}^{2(1)} & g_{zy}^{1(-1)} & g_{zy}^{2(-1)} \\ g_{zz}^{1(0)} & g_{zz}^{2(0)} & g_{zz}^{1(1)} & g_{zz}^{2(1)} & g_{zz}^{1(-1)} & g_{zz}^{2(-1)} \end{pmatrix} \neq 0, \tag{33}$$

we have that all the elements

$$h_k \otimes \sigma_\alpha^1\sigma_\beta^2 \in \mathcal{L}, \qquad x_k \otimes \sigma_\alpha^1\sigma_\beta^2 \in \mathcal{L}, \qquad y_k \otimes \sigma_\alpha^1\sigma_\beta^2 \in \mathcal{L}, \tag{34}$$

where $k = 1, 2$ and $\alpha, \beta = x, y, z$.

So we need further to prove $1 \otimes \sigma_\alpha^1\sigma_\beta^2 \in \mathcal{L}$ and $h_k \otimes 1 \in \mathcal{L}$, $x_k \otimes 1 \in \mathcal{L}$, $y_k \otimes 1 \in \mathcal{L}$. To this end let us evaluate

$$-\tfrac{1}{2}\left[\mathrm{i}h_k \otimes \sigma_\alpha^1\sigma_\beta^2, \mathrm{i}x_k \otimes \sigma_\alpha^1\sigma_\beta^2\right] = y_k \otimes 1 \in \mathcal{L}, \tag{35}$$

$$\tfrac{1}{2}\left[\mathrm{i}h_k \otimes \sigma_\alpha^1\sigma_\beta^2, y_k \otimes \sigma_\alpha^1\sigma_\beta^2\right] = \mathrm{i}x_k \otimes 1 \in \mathcal{L}, \tag{36}$$

$$\tfrac{1}{2}\left[\mathrm{i}x_k \otimes \sigma_\alpha^1\sigma_\beta^2, y_k \otimes \sigma_\alpha^1\sigma_\beta^2\right] = \mathrm{i}h_k \otimes 1 \in \mathcal{L}. \tag{37}$$

In this three-dimensional system case, we choose all coefficients $g_{x\alpha}^{j(k)} = 0$. Then from $\mathrm{i}H_0''$ we subtract the Lie algebra elements (25)–(27), (30)–(32) with the same accessor part $\sigma_\alpha^1\sigma_\beta^2$ $(\alpha, \beta = x, y, z)$ and find

$$\mathrm{i}1 \otimes \sigma_x^1\sigma_x^2 \in \mathcal{L} \tag{38}$$

from which we have $\mathrm{i}1 \otimes \sigma_\alpha^1\sigma_\beta^2 \in \mathcal{L}$. So we have proved the complete controllability of three-dimensional quantum systems.

Here we would like to note that the Lie algebra su(3) has 6 Chevalley basis and therefore we need six equations to decouple the terms in Hamiltonian. However, there are nine elements of type $\mathrm{i}\sigma_\alpha^1\sigma_\beta^2$.

5. Complete controllability of a finite-dimensional quantum system

With experience built in the previous two sections, we shall generally investigate the complete controllability of arbitrary finite-dimensional quantum systems in this section. In the interaction Hamiltonian H_I there are 3^M coupling terms

$$\left[\sum_{j=1}^{N-1}\sum_k g_{\{\alpha_i\}}^{j(k)} s_j^k\right] \otimes \sigma_{\alpha_1}^1\sigma_{\alpha_2}^2\cdots\sigma_{\alpha_M}^M. \tag{39}$$

Here we call them *nomial* for convenience. Note that the accessor part in each nomial is labeled by an index set $\{\alpha_i\}$. In the forthcoming part of this paper we use symbol $\{\alpha_i|n\}$ to denote this index set, in which the number of σ_z in the nomials is not less than n. It is obvious for $\{\alpha_i|n\}$, there are

$$\binom{M}{n} 2^{M-n} \tag{40}$$

nomials in which there are $n\sigma_z$'s. One can easily check that the sum of those numbers gives rise to 3^M using the binomial formula, the total coupling terms in H_I, as we expected.

In the following we shall first prove each nomials (39) is in Lie algebra $\mathcal{L}$ and then prove the generated Lie algebra is su$(N2^M)$.

5.1. Decoupling $\mathrm{i}H_I$ *to nomials*

We first prove that each terms in H_I is in the Lie algebra $\mathcal{L}$. We shall prove this recursively according to the number of σ_z in each nomial.

We first note that the element $\mathrm{i}H_A^0 \in \mathcal{L}$, so the element $\mathrm{i}H^{(0)} \equiv \mathrm{i}H_0 - \mathrm{i}H_A^0 \in \mathcal{L}$. Without losing generality, we suppose $M \geqslant 3$ hereafter.

As the first step, we would like to *select* the terms without σ_z in the qubit chain. For this purpose, we first annihilate $\mathrm{i}H_A^I$ and the nomials with σ_z's in $\mathrm{i}H_I$ from $\mathrm{i}H^0$ by evaluating the commutation relations

$$\begin{aligned}\mathrm{i}H^{(0)1} &\equiv \left[\mathrm{i}\sigma_z^M, \left[\mathrm{i}\sigma_z^{M-1}, \ldots, \left[\mathrm{i}\sigma_z^1, \mathrm{i}H^{(0)}\right]\cdots\right]\right] \\ &= \mathrm{i}2^M \sum_{[\![\alpha_i]\!]} (-1)^{\Delta_{[\![\alpha_i]\!]}} \left[\sum_{j=1}^{N-1}\sum_k g_{[\![\alpha_i]\!]}^{j(k)} s_j^k\right] \otimes \sigma_{\bar{\alpha}_1}^1\sigma_{\bar{\alpha}_2}^2\cdots\sigma_{\bar{\alpha}_M}^M \in \mathcal{L},\end{aligned} \tag{41}$$

where we have used the symbol $[\![\alpha_i]\!]$ to denote the index set with each $\alpha_i = x, y$ only, and $\Delta_{[\![\alpha_i]\!]}$ is the number of x in $[\![\alpha_i]\!]$. Note that the terms $\mathrm{i}H_A^I$ are also annihilated as each term in it has only two neighborhood qubits.

As each index in (41) is either x or y, we can use selection operators to pick up each nomial in equation (41)

$$S^M_{\beta_M,\overline{\beta}_M} S^{M-1}_{\beta_{M-1},\overline{\beta}_{M-1}} \cdots S^1_{\beta_1,\overline{\beta}_1} (\mathrm{i}H^{(0)1}) = \mathrm{i}\left[\sum_{j=1}^{N-1}\sum_k g^{j(k)}_{\{\beta_i\}} s^k_j\right] \otimes \sigma^1_{\overline{\beta}_1}\sigma^2_{\overline{\beta}_2}\cdots\sigma^M_{\overline{\beta}_M} \in \mathcal{L}, \tag{42}$$

where $\beta_i, \overline{\beta}_i = x, y$ and

$$\overline{\beta}_i = \begin{cases} x, & \text{if } \beta_i = y; \\ y, & \text{if } \beta_i = x. \end{cases} \tag{43}$$

So equation (42) implies that we have 2^M Lie algebra elements in which β_i is either x or y.

As the second step, we further deprive nomials with just one σ_z in the qubit chain. We first evaluate the proper commutation with external interaction Hamiltonian to change $\sigma^k_{\overline{\beta}_k}$ to $\sigma^k_{\beta_k}$ in equation (42) and then subtract them from $\mathrm{i}H^{(0)}$. We obtain the following Lie algebra element:

$$\mathrm{i}H^{(1)} \equiv \mathrm{i}H_S + \mathrm{i}H_A^I + \mathrm{i}\sum_{\{\alpha_i|1\}}\left[\sum_{j=1}^{N-1}\sum_k g^{j(k)}_{\{\alpha_i|1\}} s^k_j\right] \otimes \sigma^1_{\alpha_1}\sigma^2_{\alpha_2}\cdots\sigma^M_{\alpha_M} \in \mathcal{L}. \tag{44}$$

Without losing generality, we consider the case where $\alpha_1 = z$. As in step 1, we would like to annihilate the term $\mathrm{i}H_A^I$ and all nomials that has σ_z in sites other than site 1. For this purpose, we evaluate the commutation relation on the site 1 with $\mathrm{i}\sigma^1_x$ and other sites with σ^j_z. Those M operations change σ^1_z to σ^1_y and other σ^n_x to σ^n_y or vise versa for $2 \leqslant n \leqslant M$. We have

$$\begin{aligned} \mathrm{i}H^{(1)1} &\equiv \left[\mathrm{i}\sigma^M_z, \cdots, \left[\mathrm{i}\sigma^2_z, \left[\mathrm{i}\sigma^1_x, \mathrm{i}H^{(1)}\right]\cdots\right]\right] \\ &= \mathrm{i}2^M \sum_{\substack{\{\alpha_i|1\}\\ \alpha_1=z}} (-1)^{\Delta_{\{\alpha_i|1\}}} \left[\sum_{j=1}^{N-1}\sum_k g^{j(k)}_{\{\alpha_i|1\}} s^k_j\right] \otimes \sigma^1_y\sigma^2_{\overline{\alpha}_2}\cdots\sigma^M_{\overline{\alpha}_M} \in \mathcal{L}. \end{aligned} \tag{45}$$

where $\alpha_i = x$ or y for $i = 2, 3, \ldots, M$. As each site of the accessor is either σ_x or σ_y, we can use selection operators to find 2^{M-1} elements of $\mathcal{L}$

$$\mathrm{i}\left[\sum_{j=1}^{N-1}\sum_k g^{j(k)}_{\{z,\beta_2,\ldots,\beta_M|1\}} s^k_j\right] \otimes \sigma^1_y\sigma^2_{\overline{\beta}_2}\cdots\sigma^M_{\overline{\beta}_M} \in \mathcal{L}. \tag{46}$$

In fact, we can use the same method to prove that nomials with only one $\alpha_k = z$ on the site k are elements of the Lie algebra $\mathcal{L}$. In total we have $M2^{M-1}$ such type Lie algebra elements.

We can now subtract the element

$$\mathrm{i}\left[\sum_{j=1}^{N-1}\sum_k g^{j(k)}_{\{z,\beta_2,\ldots,\beta_M|1\}} s^k_j\right] \otimes \sigma^1_z\sigma^2_{\beta_2}\cdots\sigma^M_{\beta_M} \in \mathcal{L} \tag{47}$$

which can be obtained from the commutation relation of σ^1_x with (46), and obtain an element of $\mathcal{L}$ which takes the same form of (44) but there are at least two z in $[\![\alpha_i]\!]$

$$\mathrm{i}H^{(2)} \equiv \mathrm{i}H_S + \mathrm{i}H_A^I + \mathrm{i}\sum_{\{\alpha_i|2\}}\left[\sum_{j=1}^{N-1}\sum_k g^{j(k)}_{\{\alpha_i|2\}} s^k_j\right] \otimes \sigma^1_{\alpha_1}\sigma^2_{\alpha_2}\cdots\sigma^M_{\alpha_M} \in \mathcal{L}. \tag{48}$$

Suppose that $\alpha_m = \alpha_n = z\ (m \neq n)$. Then we can evaluate the commutation relation of $\mathrm{i}H^{(2)}$ with $\mathrm{i}\sigma_x^m$, $\mathrm{i}\sigma_x^n$ and $\mathrm{i}\sigma_z^k$ $(k \neq m, n)$, one can easily prove that the element with two z in the Lie algebra $\mathcal{L}$.

Following the procedure recursively on the number of z in $\{\alpha_i\}$, we can prove all the elements

$$\mathrm{i}\left[\sum_{j=1}^{N-1}\sum_{k} g_{\{\alpha_i\}}^{j(k)} s_j^k\right] \otimes \sigma_{\alpha_1}^1 \sigma_{\alpha_2}^2 \cdots \sigma_{\alpha_M}^M \in \mathcal{L}, \tag{49}$$

where each $\alpha_i = x, y, z$. There are 3^M such elements.

As each nomial of the type (49) is a linear combination of $3(N-1)$ elements x_i, y_i and h_i $(i = 1, 2, \ldots, N-1)$, so we require that the number of qubits is big enough such that

$$3^M \geqslant 3(N-1), \tag{50}$$

and then choose $3(N-1)$ elements of type (49). Then we further require the determinant of the coefficient matrix

$$\det\left(g_{\{\alpha_i\}}^{i(k)}\right) \neq 0, \tag{51}$$

we find that the elements

$$\mathrm{i}x_i \otimes \sigma_{\alpha_1}^1 \sigma_{\alpha_2}^2 \cdots \sigma_{\alpha_M}^M, \tag{52}$$

$$\mathrm{i}y_i \otimes \sigma_{\alpha_1}^1 \sigma_{\alpha_2}^2 \cdots \sigma_{\alpha_M}^M, \tag{53}$$

$$\mathrm{i}h_i \otimes \sigma_{\alpha_1}^1 \sigma_{\alpha_2}^2 \cdots \sigma_{\alpha_M}^M \tag{54}$$

are Lie algebra elements. Namely, all nomials in interaction Hamiltonian $\mathrm{i}H_I$ are decoupled and each term is in the Lie algebra $\mathcal{L}$.

5.2. System operators as Lie algebra elements

It is easy to see that

$$\tfrac{1}{2}\left[\mathrm{i}x_i \otimes \sigma_{\alpha_1}^1 \sigma_{\alpha_2}^2 \cdots \sigma_{\alpha_M}^M, y_i \otimes \sigma_{\alpha_1}^1 \sigma_{\alpha_2}^2 \cdots \sigma_{\alpha_M}^M\right] = \mathrm{i}h_i \otimes 1_A \in \mathcal{L}, \tag{55}$$

$$-\tfrac{1}{2}\left[\mathrm{i}h_i \otimes \sigma_{\alpha_1}^1 \sigma_{\alpha_2}^2 \cdots \sigma_{\alpha_M}^M, \mathrm{i}x_i \otimes \sigma_{\alpha_1}^1 \sigma_{\alpha_2}^2 \cdots \sigma_{\alpha_M}^M\right] = y_i \otimes 1_A \in \mathcal{L}, \tag{56}$$

$$-\tfrac{1}{2}\left[\mathrm{i}h_i \otimes \sigma_{\alpha_1}^1 \sigma_{\alpha_2}^2 \cdots \sigma_{\alpha_M}^M, \mathrm{i}y_i \otimes \sigma_{\alpha_1}^1 \sigma_{\alpha_2}^2 \cdots \sigma_{\alpha_M}^M\right] = \mathrm{i}y_i \otimes 1_A \in \mathcal{L}. \tag{57}$$

From those Chevalley basis elements corresponding to simple roots of Lie algebra su(*N*), we can further construct the standard Cartan basis of su(*N*) corresponding any other positive and negative roots. We have in total $N^2 - 1$ such basis elements of $\mathcal{L}$.

5.3. Accessor elements

Above discussions mean that the Hamiltonian $\mathrm{i}H_S^0$ is an element of $\mathcal{L}$. So subtracting this element along with $\mathrm{i}H_I$ and $\mathrm{i}H_A^0$, we find that $\mathrm{i}H_A^I \in \mathcal{L}$.

It is easy to see that

$$\left[\left[\mathrm{i}H_A^I, \mathrm{i}1 \otimes \sigma_y^1\right], \mathrm{i}1 \otimes \sigma_y^1\right] = -\mathrm{i}4c_1\left(1_S \otimes \sigma_x^1 \sigma_x^2\right) \in \mathcal{L} \tag{58}$$

thanks to the condition $c_1 \neq 0$. We further have that

$$\left[\left[\mathrm{i}H_A^I - \mathrm{i}c_1 1_S \otimes \sigma_x^1 \sigma_x^2, \mathrm{i}1 \otimes \sigma_y^2\right], \mathrm{i}1 \otimes \sigma_y^2\right] = -\mathrm{i}4c_2\left(1_S \otimes \sigma_x^2 \sigma_x^3\right) \in \mathcal{L} \tag{59}$$

since $c_2 \neq 0$. Repeating this process we can prove that

$$\mathrm{i}\big(1_S \otimes \sigma_x^j \sigma_x^{j+1}\big) \in \mathcal{L}, \qquad j = 1, 2, \ldots, M-1. \tag{60}$$

Then form lemma 2 in [17], we find that

$$\mathrm{i}(1_S \otimes \sigma_{[\alpha]}) \in \mathcal{L}, \qquad [\alpha] \neq (0, 0, \ldots, 0). \tag{61}$$

The number of those type of elements is $4^M - 1$.

5.4. Complete controllability

So far we have proved that if the conditions (50) and (51) are satisfied, the following elements are in Lie algebra $\mathcal{L}$:

$$\begin{aligned} &h_i \otimes 1_A \in \mathcal{L}, \qquad \mathrm{i}x_i \otimes 1_A \in \mathcal{L}, \qquad \mathrm{i}y_i \otimes 1_A \in \mathcal{L}, \\ &h_i \otimes \sigma_{[\alpha]} \in \mathcal{L}, \qquad \mathrm{i}x_i \otimes \sigma_{[\alpha]} \in \mathcal{L}, \qquad \mathrm{i}y_i \otimes \sigma_{[\alpha]} \in \mathcal{L}, \\ &\mathrm{i}(1_S \otimes \sigma_{[\alpha]}) \in \mathcal{L}, \end{aligned} \tag{62}$$

and their corresponding Cartan basis elements. The total number of those Lie algebra elements is

$$(N^2 - 1) + (N^2 - 1)(4^M - 1) + (4^M - 1) = (2^M N)^2 - 1, \tag{63}$$

which is the dimension of Lie algebra $\mathrm{su}(2^M N)$. This proves the complete controllability of the indirect control system (7).

6. Conclusion

In this paper we have proposed a scheme for the indirect control of finite-dimensional quantum systems via the quantum accessor modeled as a qubit chain with XY-type coupling. The main results of this paper are as follows:

- Different from our previous paper [17], we do not need to apply an excitation classical field on the controlled system. So this scheme is a *pure* indirect control in the sense that the classical control fields control the accessor only.
- The minimal number M for the complete control of the controlled system is determined by condition (50), while in previous scheme [17], the minimal M is determined by $2^M \geqslant 2(N-1)$. It is obvious that the scheme proposed here requires less qubits in accessor in comparison to the proposal [17].
- We also note that in the process of decoupling the interaction Hamiltonian (see section 5.1) we do not put any requirements on the structure of the energy level of the controlled system, while in [17], the indirect controllability reduces to the semi-classical control investigated in [9, 10] which depends on the energy-level structure of the controlled system.
- From equation (51) we find that the controllability is determined by the way of coupling of the controlled system and the accessor. So in a practical control protocol we can design a simplest coupling of the controlled system and the accessor to ensure the complete control of the controlled system, according to the condition (51).

Therefore we believe the scheme in this paper has wider applicability. As further works we would like to study the concrete control protocol of the indirect control, and examine the graph connectivity for assessing the controllability of quantum systems, as well as applications in quantum information processing.

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